

# Closed-form analytic solutions for dilogarithmic double integrals

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**Abstract.** This note presents techniques to analytically solve double integrals of the dilogarithmic type which are of great importance in the perturbative treatment of quantum-field theory. In our approach divergent integrals can be calculated similar to their convergent counterparts after identifying and isolating their singular parts.

**Keywords:** integration method, dilogarithmic integral

**1. Introduction and physical motivation.** In quantum field theory the differential volume element for the three-particle phase space in four-dimensional Minkowski space has the form (neglecting overall normalization and statistical factors) of a product containing the relativistic four-momenta  $p_i$  of the three particles:

$$d\text{PS}_3 \sim \prod_{i=1}^3 d^4 p_i. \quad (1)$$

Usually one includes energy-momentum conservation ( $q = \sum_{i=1}^3 p_i$ ) and the on-shell requirement for the particles ( $p_i^2 = m_i^2$ ) as Dirac delta functions with positive-energy solutions so that

$$d\text{PS}_3 \sim \prod_{i=1}^3 d^4 p_i \delta_+(p_i^2 - m_i^2) \delta^4\left(q - \sum_{i=1}^3 p_i\right), \quad (2)$$

where  $q$  is the total momentum of the reaction and  $m_i$  are the masses of the final particles under consideration. This expression is known as the Lorentz invariant three-body phase space and its integration over the kinematic  $S$ -matrix yields numerical predictions for the measurable cross sections in physical experiments (see *e.g.* [1] and [2]).

The general phase space Eq. (2) has 5 degrees of freedom, resulting from 7 restrictions (3 by the on-shell requirement and 4 by energy-momentum conservation) on the overall 12 components of the four-momenta. Integrating out all dependence of the undetected spatial components of momentum  $p_3$ , leaves only two degrees of freedom to parameterize the phase space. Therefore, a convenient choice to describe the phase space are the two Lorentz invariants [3]:

$$y = 1 - \frac{2p_1 \cdot q}{q^2} \quad \text{and} \quad z = 1 - \frac{2p_2 \cdot q}{q^2}. \quad (3)$$

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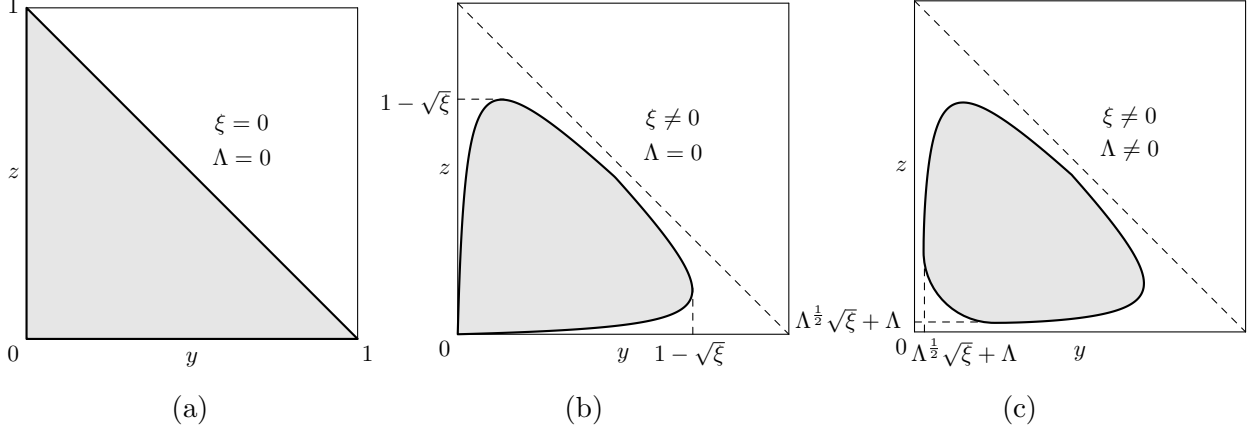


FIGURE 1. Three-particle phase space.

With this parameterization the three-particle phase space simplifies considerably, namely one finally obtains

$$d\text{PS}_3 \sim dy dz, \quad (4)$$

and the integration limits introduced in [3] are

$$\begin{cases} y_+ &= 1 - \sqrt{\xi} \\ y_- &= \Lambda^{1/2} \sqrt{\xi} + \Lambda \end{cases} \quad (5a)$$

$$z_{\pm} = \frac{2y}{4y + \xi} \left[ 1 - y - \frac{1}{2}\xi + \Lambda + \frac{\Lambda}{y} \pm \frac{1}{y} \sqrt{(1-y)^2 - \xi} \sqrt{(y-\Lambda)^2 - \Lambda\xi} \right], \quad (5b)$$

where the masses squared are denoted as  $\xi = 4m_i^2/q^2$  with  $m_i = m_1 = m_2$ , and  $\Lambda = m_3^2/q^2$ . Hence, the cross sections reduce to the following structure

$$\int d\text{PS}_3 \sum_i f_i \sim \sum_i \int dy dz f_i(y, z), \quad (6)$$

where the individual  $f_i(y, z)$  are all simple rational functions in  $y$  and  $z$ . We have found that if differential and doubly polarized states are considered [4], these double integrals generalize to the types  $(m, n)$

$$\int dy dz \frac{f_i(y, z)}{[(1-y)^2 - \xi]^{m/2} [(1-z)^2 - \xi]^{n/2}}, \quad (7)$$

where  $m, n = 0, 1, 2, 3$  independently. The integrals of (7) will generally display two classes of divergences. The so-called *collinear* singularities manifest themselves as poles and/or logarithmic singularities in  $\xi = 0$  and occur on the boundaries of the triangle in Figure 1a. They can be *regularized* by assuming  $\xi \neq 0$ , which deforms the phase space according to Figure 1b. Here, the so-called *soft* divergences still survive at the origin and take the form of a logarithmic singularity in  $\Lambda$ . Assuming  $\Lambda \neq 0$  finally renders all intermediate phase-space integrals singularity-free. It is important to notice that the final cross section (6) is a physical quantity and can not contain any divergences. Therefore, in the total sum the limits  $\xi \rightarrow 0$  and  $\Lambda \rightarrow 0$  will always yield finite results. The technique of *regularization* is a means of controlling the intermediate divergences caused by

the particular decomposition (7). Typical three-body reactions in quantum field theory consider two massive particles and one massless particle, assuming  $\xi \neq 0$  and  $\Lambda = 0$ .

**2. Convergent integrals.** For the convergent case, we use as a demonstration of our technique the simplest integral of the type (2, 1):

$$I = \int \frac{dy dz}{[(1-y)^2 - \xi] \sqrt{(1-z)^2 - \xi}} = \int_{y_-}^{y_+} \frac{dy}{(1-y)^2 - \xi} \ln \left( \sqrt{(1-z)^2 - \xi} + z - 1 \right) \Big|_{z_-}^{z_+}, \quad (8)$$

which contains the phase-space boundaries (5) with  $\Lambda = 0$ . There is no pole at  $(1-y)^2 - \xi = 0$  as the individual contributions  $I_+$  and  $I_-$ , corresponding to the limits  $z_+$  and  $z_-$ , have similar asymptotic behavior with opposite signs and cancel each other. However, for a separate calculation of these parts, we require a spurious cut  $0 < \varepsilon \ll 1$ :

$$I = I_+ + I_- \quad \text{with} \quad I_{\pm} = \pm \int_0^{1-\sqrt{\xi}-\varepsilon^2} \frac{dy}{(1-y)^2 - \xi} \ln \left( \sqrt{(1-z_{\pm})^2 - \xi} + z_{\pm} - 1 \right). \quad (9)$$

Next, we consider the substitutions

$$k_{\pm} = 1 - y \pm \sqrt{(1-y)^2 - \xi} \quad (10a)$$

for  $I_{\pm}$ , respectively, which gives the following integration limits at the singular points:

$$k_{\pm} = \sqrt{\xi} \pm \tilde{\varepsilon} + \mathcal{O}(\mathfrak{E}) \quad \text{with} \quad \mathfrak{E} = \sqrt{\varepsilon} \xi^{\frac{\infty}{2}} \varepsilon. \quad (10b)$$

Applying (10) in (9) and using the new variable  $v = \sqrt{1-\xi}$  in the limits, one gets

$$I_+ = \frac{1}{\sqrt{\xi}} \int_{\sqrt{\xi+\tilde{\varepsilon}}}^{2-\sqrt{\xi}} dk \left[ \frac{1}{k-\sqrt{\xi}} - \frac{1}{k+\sqrt{\xi}} \right] \left\{ \ln \xi - \ln(2-k) \right\} \\ + \frac{1}{\sqrt{\xi}} \int_{2-\sqrt{\xi}}^{1+v} dk \left[ \frac{1}{k-\sqrt{\xi}} - \frac{1}{k+\sqrt{\xi}} \right] \ln(2-k), \quad (11a)$$

$$I_- = \frac{1}{\sqrt{\xi}} \int_{1-v}^{\sqrt{\xi}-\tilde{\varepsilon}} dk \left[ \frac{1}{k-\sqrt{\xi}} - \frac{1}{k+\sqrt{\xi}} \right] \left\{ \ln \xi - \ln(2-k) \right\}. \quad (11b)$$

Integrals (11) can now be solved without further difficulty in terms of real dilogarithms [5]

$$\text{Li}_2(x) = - \int_0^x d\rho \frac{\ln(1-\rho)}{\rho}. \quad (12)$$

Especially for more complicated integrals of the general type  $(m, n)$ , we use repeated cyclic application of dilogarithmic identities (see also [5]) to dramatically simplify the final solutions. The final result is:

$$\begin{aligned}
I = & 2 \operatorname{Li}_2 \left( \frac{\sqrt{\xi}}{2 + \sqrt{\xi}} \right) - 2 \operatorname{Li}_2 \left( \frac{\sqrt{\xi}}{2 - \sqrt{\xi}} \right) - \operatorname{Li}_2 \left( \frac{1 - v}{2 + \sqrt{\xi}} \right) + \operatorname{Li}_2 \left( \frac{1 - v}{2 - \sqrt{\xi}} \right) - \operatorname{Li}_2 \left( \frac{2 - \sqrt{\xi}}{1 + v} \right) \\
& - \operatorname{Li}_2 \left( \frac{1 + v}{2 + \sqrt{\xi}} \right) + \frac{1}{4} \ln \left( \frac{1 + v}{1 - v} \right) \left[ -\ln \xi + 2 \ln(2 - \sqrt{\xi}) - \frac{1}{2} \ln \left( \frac{1 + v}{1 - v} \right) \right] \\
& + \frac{1}{2} \ln \xi \left[ -\frac{1}{4} \ln \xi + \ln(2 - \sqrt{\xi}) \right] - \frac{1}{2} \ln^2(2 - \sqrt{\xi}) + \frac{\pi^2}{3}.
\end{aligned} \tag{13}$$

It shall be noted, that rigorous numerical tests have confirmed the validity of our method. Full analytic results of integrals of this type have been applied in various contexts (see *e.g.* Refs. [3,4] and work in progress).

**3. Divergent integrals.** For some pathological cases the intermediate phase-space integrals will diverge because of additional poles in the integrand functions  $f_i$  in (7). Then, one needs to regularize with  $\Lambda \neq 0$  to avoid contact with the origin as shown in Figure 1c. A more intricate example is  $f = 1/z^2$  for integral type  $(2, 1)$ :

$$J = \int \frac{dy dz}{[(1 - y)^2 - \xi] \sqrt{(1 - z)^2 - \xi}} \frac{1}{z^2} \tag{14}$$

Because of the phase-space symmetry  $y \leftrightarrow z$ , it is easy to see that this integral takes the form

$$J = -\frac{1}{2\sqrt{\xi}} \int_{y_-}^{y_+} \frac{dy}{\sqrt{(1 - y)^2 - \xi}} \frac{1}{y^2} \ln \left( -\frac{z - 1 + \sqrt{\xi}}{z - 1 - \sqrt{\xi}} \right) \Big|_{z_-}^{z_+} \tag{15}$$

with the complicated phase-space boundaries (5) for  $\Lambda \neq 0$ . For  $|z| \ll 1$ , we use the expansion

$$\ln \left( -\frac{z - 1 + \sqrt{\xi}}{z - 1 - \sqrt{\xi}} \right) = \ln \left( -\frac{1 - \sqrt{\xi}}{1 + \sqrt{\xi}} \right) - \frac{2\sqrt{\xi}}{v^2} z + \mathcal{O}(\dagger^\epsilon) \tag{16}$$

to split the integral into

$$J = \underbrace{\frac{1}{v^2} \int \frac{dy dz}{\sqrt{(1 - y)^2 - \xi}} \frac{1}{y^2}}_{J_1} + \underbrace{\int_{y_-}^{y_+} \frac{dy}{\sqrt{(1 - y)^2 - \xi}} \frac{1}{y^2} \left[ -\frac{1}{2\sqrt{\xi}} \ln \left( -\frac{z - 1 + \sqrt{\xi}}{z - 1 - \sqrt{\xi}} \right) - \frac{1}{v^2} z \right]_{z_-}^{z_+}}_{J_2}, \tag{17}$$

where the second integral  $J_2$  is rendered convergent. Therefore, we can safely put  $\Lambda = 0$  in this term and proceed as outlined in Section 2 (with *two* spurious cuts, one for the upper and one for the lower integration limit). Straightforward computation yields:

$$J_2 = \operatorname{Li}_2 \left( -\sqrt{\frac{1 + v}{1 - v}} \right) - \operatorname{Li}_2 \left( -\sqrt{\frac{1 - v}{1 + v}} \right) + 2 \left[ \operatorname{Li}_2 \left( \sqrt{\frac{1 - v}{1 + v}} \right) + \operatorname{Li}_2 \left( 2 \frac{1 - \sqrt{\xi}}{1 + v - \sqrt{\xi}} \right) \right]$$

$$\begin{aligned}
& -\text{Li}_2\left(2\frac{1-\sqrt{\xi}}{1-v-\sqrt{\xi}}\right) - \text{Li}_2\left(\frac{2}{1+v+\sqrt{\xi}}\right) - \text{Li}_2\left(\frac{1-v+\sqrt{\xi}}{2}\right) \Big] \\
& + \frac{1}{8} \left[ \ln \xi - 2 \ln(1-\sqrt{\xi}) - 12 \ln v + \frac{1}{2} \ln\left(\frac{1+v}{1-v}\right) - 2 \ln 2 \right] \ln\left(\frac{1+v}{1-v}\right) \\
& + \frac{1}{4} \left[ -\frac{1}{4} \ln \xi - \ln(1+\sqrt{\xi}) + \frac{8v}{\xi} + \ln 2 \right] \ln \xi - \frac{1}{2} \left[ \frac{1}{2} \ln(1+\sqrt{\xi}) - \ln 2 \right] \ln(1+\sqrt{\xi}) \\
& + \frac{4v}{\xi} \left[ -\ln \xi + 2 \ln(2-\sqrt{\xi}) + \frac{1}{2} \ln\left(\frac{1+\sqrt{\xi}}{1-\sqrt{\xi}}\right) - \ln 2 + 1 \right] - \frac{1}{4} \ln^2 2 + \frac{\pi^2}{3}. \tag{18}
\end{aligned}$$

The splitting procedure (17) works for all known cases using adequate expansions similar to (16). Its main objective is to isolate the logarithmic divergences in  $\Lambda$  and reduce the difficult divergent part of the integrals to integrals of lower type. Here, the first integral of (17) is of the lower type (1,0):

$$J_1 = \int \frac{dy dz}{\sqrt{(1-y)^2 - \xi}} \frac{1}{y^2}. \tag{19}$$

By integrating over  $z$  and using (5b), it follows that

$$J_1 = \int \frac{dy}{\sqrt{(1-y)^2 - \xi}} \frac{1}{y^2} (z_+ - z_-) = 4 \int_{\Lambda^{\frac{1}{2}}\sqrt{\xi}+\Lambda}^{1-\sqrt{\xi}} \frac{dy}{y^2} \frac{\sqrt{(y-\Lambda)^2 - \Lambda\xi}}{4y + \xi}, \tag{20}$$

which again can be solved by adding and subtracting a splitting term to the integrand. The appropriate splitting term is  $\sqrt{(y-\Lambda)^2 - \Lambda\xi}/\xi$ . By neglecting finite pieces in  $\Lambda$ , one obtains

$$J_1 = \frac{4}{\xi} \int_{\Lambda^{\frac{1}{2}}\sqrt{\xi}+\Lambda}^{1-\sqrt{\xi}} \frac{dy}{y^2} \sqrt{(y-\Lambda)^2 - \Lambda\xi} + 4 \int_0^{1-\sqrt{\xi}} \frac{dy}{y} \left[ \frac{1}{4y + \xi} - \frac{1}{\xi} \right] + \mathcal{O}(*). \tag{21}$$

The integral form (21) can now be solved by standard integration techniques (*e.g.* letting  $t = y - \Lambda$  in the first integral). Further expansion in  $\Lambda$  finally produces

$$J_1 = \frac{4}{\xi} \left[ -\ln \Lambda^{\frac{1}{2}} + \frac{1}{2} \ln \xi + \ln 2 + \ln(1-\sqrt{\xi}) - 2 \ln(2-\sqrt{\xi}) - 1 + \mathcal{O}(*) \right] \tag{22}$$

so that  $J = J_1 + J_2$  in (14) is fully determined.

**4. Conclusion.** We have outlined a general integration method to obtain closed-form analytic solutions for complicated dilogarithmic double integrals that emerge in applications such as perturbative quantum field theory. Our method combines suitable substitutions, repeated splitting and isolation of divergent integral parts, and cyclic usage of dilogarithmic identities.

In particular, with this approach it was possible to calculate fully analytic predictions for reactions in particle physics [3,4], where in the past only numerical estimates were available. It also permitted to find simple and highly accurate polynomial approximations of the Schwinger type for these processes [4].

The method has been successfully implemented on computer algebra systems to help create extensive tables of relevant integrals of type  $(m, n)$ . They form the basis of numerical C/FORTRAN libraries that will be available in the near future.

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